GALOIS THEORY TOPIC I
CONSTRUCTIBILITY

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Abstract. We discuss the classical Greek notion of constructibility of geometric objects. The reader is invited to obtain a ruler and compass to perform the exercises and follow the constructions described in the proofs.

1. Construction with Straight-Edge and Compass

The drawings of the ancient Greek geometers were made using two instruments: a straight-edge and a compass.

A *straight-edge* draws lines. With the straightedge, we are permitted to draw a straight line of indefinite length through any two given distinct points. The straight-edge is unmarked; it cannot measure distances.

A *compass* draws circles. With the compass, we are permitted to draw a circle with any given point as the center and passing through any given second point. The compass collapses if it is lifted; we are not *a priori* permitted to use it to measure the distance between given points, and draw a circle around another given point of the same radius.

The straight-edge and the compass have come to be known as *Euclidean tools*, although the quest to construct points using them pre-dates Euclid by two centuries.

2. The Three Greek Problems

As the Greeks investigated what could be accomplished with their Euclidean tools, three interesting unsolved problems arose.

Problem 1 (Duplication of the Cube)**.** Given a cube, construct a cube with double the volume.

Problem 2 (Trisection of an Angle)**.** Given an angle, construct an angle one third as large.

Problem 3 (Quadrature of the Circle)**.** Given a circle, construct a square with the same area.

We now attempt to make the statements of these problems precise, using modern notation.

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3. Construction of Points in a Plane

Let P denote the set of all points in a plane. Let $Q \subset P$.

Let $\mathcal{L}(Q)$ denote the set of all lines in P which pass through at least two points in Q; these are the *lines constructible from Q in one stage*. Let $\mathcal{K}(Q)$ denote the set of all circles in P whose center is a point in Q which pass through at least one point in Q; these are the *circles constructible from Q in one stage*. Let $\mathcal{O}(Q) = \mathcal{L}(Q) \cup \mathcal{K}(Q)$; these are the *objects constructible from Q in one stage*. Let

$$\mathcal{C}(Q) = \{A \in P \mid \text{ there exists } O_1, O_2 \in \mathcal{O}(Q) \text{ such that } A \in O_1 \cap O_2\} \cup Q.$$

These are the *points constructible from Q in one stage*.

Now let $\mathcal{C}_0(Q) = Q$, and for $i \geq 1$ inductively define

$$\mathcal{C}_i(Q) = \mathcal{C}(\mathcal{C}_{i-1}(Q)).$$

Then $\mathcal{C}_i(Q)$ is the set of *points constructible from Q in i stages*.

Finally, set

$$\mathcal{C}_\infty(Q) = \cup_{i=1}^{\infty} \mathcal{C}_i(Q).$$

This is the set of *points constructible from Q*. Note that if $A \in \mathcal{C}_\infty(Q)$, then $A \in \mathcal{C}_i(Q)$ for some i. We say that $A \in P$ is *constructible from Q* if $A \in \mathcal{C}_i(Q)$ for some i. Similarly, we say that a line or circle is constructible from Q if it is in $\mathcal{O}(\mathcal{C}_i(Q))$ for some i.

Exercise 1. Let P be a plane. Find the number of points which are constructible in one stage from $Q \subset P$, where Q contains

- **(a)** a single point;
- **(b)** two distinct points;
- **(c)** the vertices of an equilateral triangle;
- **(d)** an acute isosceles triangle;
- **(e)** an obtuse isosceles triangle.

Do **(d)** and **(e)** depend on the triangle chosen?

4. Standard Constructions

Let P denote a plane. For $A, B \in P$, define the following:

- AB is the line in P through A and B;
- $\overline{AB}$ is the line segment between A and B;
- $|AB|$ is the distance between A and B;
- $A - B$ is the circle through B with center A.

Also, if $C, D \in P$, then $AB \parallel CD$ represents the statement that line AB is parallel to line CD, and $AB \perp CD$ represents the statement that line AB is perpendicular to line CD.

Let Q be a set of points in the plane. We say that a line segment is constructible from Q if its endpoints are constructible from Q

Proposition 1. *Given points A and B, it is possible to construct the midpoint Z of $\overline{AB}$.*

Construction. We are given A and B.

(a) Let C and D be the points of intersection of circle $A - B$ and circle $B - A$.
(b) Let Z be the intersection of line AB and line CD.

Then Z is the midpoint of $\overline{AB}$. □

Proposition 2. *Given points A and B, it is possible to construct a point Z such that $AB \perp BZ$.*

Construction. We are given A and B.

(a) Let C be the point of intersection of line AB and circle $B - A$ which is not A.
(b) Let Z be one of the points of intersection of circle $A - C$ and circle $C - A$.

Then $AB \perp BZ$. □

Proposition 3. *Given noncolinear points A, B, and C, it is possible to construct a point Z on the line AB such that $AB \perp CZ$.*

Construction. We are given A, B, and C. If $CB \perp AB$, let $Z = C$. Otherwise, construct Z as follows.

(a) Let D be the point of intersection of line AB and circle $C - B$ which is not B.
(b) Let Z be the midpoint of $\overline{BD}$.

Then $AB \perp CZ$. □

Proposition 4. *Given noncolinear points A, B, and C, it is possible to construct a point Z such that $AB \parallel CZ$.*

Construction. We are given A, B, and C.

(a) Let D be the point of intersection of line AB and the line through C which is perpendicular to line AB.
(a) Let Z be the point of intersection of the line through A which is perpendicular to line AB and the line through C which is perpendicular to line CD.

Then $AB \parallel CZ$. □

5. Transference of Distance

Suppose we are given points A, B, and C. A *modern compass* is capable of holding its shape when lifted from the page, so that the distance between A and B can be measured using the modern compass, and then the compass is set down on C to draw a circle with center C and radius $|AB|$. We may call this process *transference of distance*. The Euclidean compass is not *a priori* capable of this feat; however, we can prove that this construction is possible.

Proposition 5. *Given noncolinear points A, B, and C, it is possible to construct a point Z such that polygon $ABCZ$ is a parallelogram.*

Construction. We have points A, B, and C.

- **(a)** Let E be the point of intersection of line BC and circle $B - A$ which lies on the C side of B.
- **(b)** Let F be the midpoint of $\overline{AE}$.
- **(c)** Let G be the point of intersection of line BF and circle $F - B$ which is not B.

Now $|AB| = |BC|$, so $\triangle ABC$ is isosceles. Moreover, F is the midpoint of the base of this isosceles triangle, so $\angle BFE$ is right, whence $\angle EFG$ is right, so $\triangle BFE \cong \triangle GFE$. Similarly, $\triangle AFB \cong \triangle AFG$; thus $|AB| = |BE| = |GE| = |GA|$. Therefore polygon $ABEG$ is a parallelogram, and in particular, line AG is parallel to line BC.

- **(e)** Let Z be the point of intersection of the line through C which is parallel to AB.

Now polygon $ABCZ$ is a parallelogram. ☐

Proposition 6 (Transference of Distance)**.** *Given points A, B, and C, it is possible to construct a point Z such that $|AB| = |CZ|$.*

Construction. Form parallelogram $ABCZ$. ☐

6. Construction of Squares

A square is constructible if its vertices are constructible.

Quadrature is the process of constructing a square whose area is equal to the area of a given plane region. A plane region with area x is called *quadrable* if it is possible to construct a square with area x. By the Proposition 2, this is equivalent to the the ability to construct a line segment of length $\sqrt{x}$.

The ancient Egyptians estimated areas of certain regions; for example they estimated that the square on 8/9 of the diameter of a circle is its quadrature. The area x of the circle with radius r would then be approximately

$$x \approx \left(\frac{8}{9}(2r)\right)^2 = \frac{256}{81}r^2;$$

this produces $\pi \approx 3.16049$.

The ancient Greeks concentrated on discovering which regions were precisely quadrable, via construction with Euclidean tools.

The third Greek problem asks if a given circle is quadrable.

7. Construction of Angles

Let P denote a plane. For $A, B, C \in P$, define the following:

- $\angle ABC$ is the angle between the line segments $\overline{AB}$ and $\overline{BC}$.

We say that an angle α is constructible from $Q \subset P$ if it is possible to construct points A, B, and C from Q such that $\alpha = \angle ABC$.

To say that an angle α is given; means that we are given points A, B, and C such that $\alpha = \angle ABC$. A *bisector* of this angle is a line BD such that $\angle ABD = \angle DBC$; then necessarily $\angle ABD = \frac{\alpha}{2}$.

Proposition 7. *Given an angle $\angle ABC$, it is possible to construct a point Z such that $\angle ABZ = \angle ZBC = \frac{\angle ABC}{2}$.*

Construction. We are given A, B, and C, with B as the vertex of the angle.

(a) Let D be the point of intersection of BC and $B - C$.

(a) Let Z be the midpoint of $\overline{CD}$.

Then $\angle ABZ = \angle ZBC$. □

Thus every given angle is *bisectable*; the second Greek problem asks if every given angle is *trisectable*.

8. Construction of Points in Space

Let S denote the set of all points in three dimensional space, and let $A, B \in S$. Although the line through A and B is well defined, there are many circles in space whose center is A which pass through B. We do not wish to say that all such circles are constructible.

We say that a plane $P \subset S$ is constructible from a set $Q \subset S$ if there exist three noncolinear points in Q which lie on P. Now circles are constructible from Q if we may construct the plane on which they lie. This gives meaning to the notion of constructibility of a point in space.

A cube is constructible if it is possible to construct its vertices in space.

The first Greek problem asks if, given a cube in space, it is possible to construct a cube in space whose volume is double that of the given cube. This is equivalent to asking if, given a line segment whose length is that of a side of the original cube, it is possible to construct a line segment whose length is that of a cube with double the volume.

9. Construction of Real Numbers

Let P be a plane and let $Q \subset P$. Let $x \in \mathbb{R}$. We say that x is constructible from Q if a line segment whose length is $|x|$ is constructible from Q. Moreover, we say simply that x is a *constructible real number* if x is constructible from $\{A, B\}$ for some $A, B \in P$ with $|AB| = 1$. Since we may consider a point to be a line segment of length 0, we consider 0 to be a constructible number.

Proposition 8. *Let $x, y \in \mathbb{R}$ be constructible. Then $x + y$ is constructible.*

Construction. Since x and y are constructible, it is possible to construct line segments of length $|x|$ and $|y|$. By Proposition 6, it is possible to construct a circle of radius $|y|$ centered at any given point.

(a) Let A and B be points such that $|AB| = |x|$.

Case 1 First assume that x and y have the same sign.

(b) Let Z be the point of intersection of line AB and the circle centered at B of radius y such that B lies on $\overline{AZ}$.

Now $\overline{AZ}$ is a line segment of length $|x| + |y| = |x + y|$.

Case 2 Next assume that x and y have different signs, and without loss of generality assume that $|x| \geq |y|$.

(b) Let Z be the point of intersection of line AB and the circle centered at B of radius y such that Z lies on $\overline{AB}$.

Now $\overline{AZ}$ is a line segment of length $|x| - |y| = |x + y|$. □

Proposition 9. *Let $x \in \mathbb{R}$ be constructible. Then $-x$ is constructible.*

Reason. This follows immediately from the definition. □

Proposition 10. *Let $x, y \in \mathbb{R}$ be constructible. Then xy is constructible.*

Construction. Since 1, x and y are constructible, it is possible to construct line segments of length 1, $|x|$, and $|y|$. Without loss of generality, we may assume that x and y are positive.

(a) Let A and B be points such that $|AB| = x$.
(b) Let C be a point of intersection of the line through A which is perpendicular to line AB and a circle centered at A of radius 1.
(c) Let D be the point of intersection line through AC and the circle centered at C of radius y such that C does not lie on $\overline{AD}$.
(d) Let Z be the intersection of line BC and the line through D which is parallel to AB.

Set $z = |DZ|$; then $\triangle CAB$ is similar to $\triangle CDZ$, so $\frac{1}{x} = \frac{y}{z}$, whence $z = xy$. $\square$

Proposition 11. *Let $x \in \mathbb{R} \setminus \{0\}$ be constructible. Then $\frac{1}{x}$ is constructible.*

Construction. Since 1 and x are constructible, it is possible to construct line segments of length 1 and $|x|$. Without loss of generality, assume that x is positive.

(a) Let A and B be points such that $|AB| = x$.
(b) Let C be the point of intersection of line AB and the circle centered at A of radius 1 such that A is not on $\overline{BC}$.
(c) Let D be a point of intersection of the line through A which is perpendicular to line AB and the circle centered at A of radius 1.
(d) Let Z be the point of intersection of line AD and the line through C which is parallel to line BD.

Set $z = |AZ|$; then $\triangle ZAC$ is similar to $\triangle DAB$, so $\frac{z}{1} = \frac{1}{x}$, that is, $z = \frac{1}{x}$. $\square$

A subset $F \subset \mathbb{R}$ with at least two elements is a *field* if it is closed under the operations of addition, subtraction, multiplication, and division. We have seen that the set of all constructible real numbers is a field. In particular, all rational numbers are constructible. Are there any others?

We show that the set of constructible numbers is closed under square roots; to do this, we need a couple of lemmas. Let's assume the geometric facts that the sum of angles in a triangle is $180°$, and that the base angles of an equilateral triangle are equal.

Lemma 1. (Thales Theorem) *An angle inscribed in a semicircle is right.*

Proof. Consider a semicircle with center O and diameter $\overline{BC}$, and let A be an arbitrary point on the semicircle; we wish to show that $\angle BAC$ is right. Now $|OA| = |OB| = |OC|$, so $\triangle BOA$ and $\triangle COA$ are isosceles triangles. Let $\alpha = \angle OBA = \angle OAB$ and $\beta = \angle OCA = \angle OAC$; then $\angle BAC = \alpha + \beta$. Adding the angles $\triangle ABC$ we obtain

$$180° = \angle OBA + \angle OCA + \angle BAC = \alpha + \beta + (\alpha + \beta) = 2(\alpha + \beta).$$

Therefore, $\angle BAC = \alpha + \beta = 90°$. □

Lemma 2. *Let $\angle ACB$ be right, and let $D \in \overline{AB}$ such that $AB \perp CD$. Then $\triangle ACB \sim \triangle ADC \sim \triangle CDB$.*

Proof. Two triangles are similar if and only if they have two equal angles. Since $\angle ACB = \angle ADC = \angle CDB = 90°$, $\angle DAC$ is shared by two of the triangles, and $\angle DBC$ is shared by two of the triangles, the result follows. □

Proposition 12. *Let $x \in \mathbb{R}$ be a constructible number. Then $\sqrt{|x|}$ is constructible.*

Construction. Since 1 and x are constructible, it is possible to construct line segments of length 1 and $|x|$. We may assume that x is positive.

- **(a)** Let A and B be points such that $|AB| = x$.
- **(b)** Let C be the point of intersection of line AB and the circle centered at B of radius 1 such that B is on $\overline{AC}$.
- **(c)** Let D be the midpoint of $\overline{AC}$.
- **(d)** Let Z be a point of intersection of the line through B which is perpendicular to line AB and the circle $D - A$.

Let $z = |BZ|$. Now $\angle ZBA = \angle ZBC = 90°$; moreover, $\angle AZC$ is right by Thales theorem. Therefore $\triangle ZBC$ is similar to $\triangle ABZ$. Thus $\frac{z}{x} = \frac{1}{z}$, whence $z^2 = x$, so $z = \sqrt{x}$. □

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